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Dynamic stability analysis of finite element modeling of piezoelectric composite plates

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Abstract

The dynamic stability of negative-velocity feedback control of piezoelectric composite plates using a finite element model is investigated. Lyapunov's energy functional based on the derived general governing equations of motion with active damping is used to carry out the stability analysis, where it is shown that the active damping matrix must be positive semi-definite to guarantee the dynamic stability. Through this formulation, it is found that imperfect collocation of piezoelectric sensor/actuator pairs is not sufficient for dynamic stability in general and that ignoring the in-plane displacements of the midplane of the composite plate with imperfectly collocated piezoelectric sensor/actuator pairs may cause significant numerical errors, leading to incorrect stability conclusions. This can be further confirmed by examining the complex eigenvalues of the transformed linear first-order state space equations of motion. To overcome the drawback of finding all the complex eigenvalues for large systems, a stable state feedback law that satisfies the second Lyapunov's stability criteria strictly is proposed. Numerical results based on a cantilevered piezoelectric composite plate show that the feedback control system with an imperfectly collocated PZT sensor/actuator pair is unstable, but asymptotic stability can be achieved by either bonding the PZT sensor/actuator pair together or changing the ply stacking sequence of the composite substrate to be symmetric. The performance of the proposed stable controller is also demonstrated. The presented stability analysis is of practical importance for effective design of asymptotically stable control systems as well as for choosing an appropriate finite element model to accurately predict the dynamic response of smart piezoelectric composite plates.

Keywords: Dynamic stability; Lyapunov stability; Piezoelectric sensors and actuators; Composite plates; Finite element method; Feedback control

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1. Introduction

Smart flexible structures often consist of thin components such as beams, plates and shells that have been fabricated from composite materials interlaced with layers of piezoelectric ceramic films. In addition to the high stiffness-to-weight ratio and strength-to-weight ratio (Reddy, 1997), the advantages of using composite materials in smart structures include the ability to tailor the structure (Matthews et al., 2000) (such as in the aero-elastic sense) and design the stiffness and strength of the laminate so as to keep the ratio of strength to weight maximum. Due to the superior mechanical properties of composite materials and the sensing and actuating capability of smart materials, the integration of composite structural design with the intelligent system design would potentially enhance the performance of smart flexible structures (Wang, 2002). This promising aspect has triggered intense research interests in composite structures bonded or embedded with piezoelectric sensors/actuators.

Exact solutions for two-dimensional analysis of a piezoelectric composite plate were presented by Ray and his co-workers (Ray et al., 1993, 1998) with exact solutions for static and dynamic analysis of a rectangular composite plate integrated with distributed piezoelectric actuators/sensors. Simple boundary conditions were considered and the displacement and the electric potential functions were initially assumed to have a sinusoidal component. For general composite structures with complicated boundary conditions, approximate numerical technique, such as the finite element (FE) method, is necessary. FE models based on Hamilton's variational principle for piezoelectric composite beams and plates have been provided by Tzou and Tseng (1990), Hwang and Park (1993), Lam et al. (1997) and Wang et al. (2001). The elements used in these works include the brick element (Tzou and Tseng, 1990), discrete Kirchhoff quadrilateral (DKQ) element (Hwang and Park, 1993), the rectangular non-conforming plate bending element (Lam et al., 1997), and isoparametric quadratic element (Wang et al., 2001). The best element to use seems to be problem dependent as each has its own advantages and disadvantages with respect to accuracy, simplicity, speed of convergence and computational cost.

In controlled structures, dynamic stability is of crucial importance in practice (Kwon and Bang, 1997). However, this issue has yet to be fully addressed in most of the models in the open literature for smart composite plates, where it is usually taken for granted that the dynamic system under consideration is asymptotically stable. In the analytical model of Hanagud et al. (1992) and the FE model of Hwang et al. (Hwang and Park, 1993; Hwang et al., 1994), in-plane displacements of the midplane have been neglected, but its consequence with respect to the dynamic stability of a negative-velocity feedback control system has not been studied. The classical negative-velocity feedback control theory requires the sensors and actuators to be collocated to guarantee the dynamic stability (Kwon and Bang, 1997), although in practice non-perfectly collocated sensors/actuators are often used to avoid the non-linearity that arises from the interactions between actuating and sensing signals (Chen et al., 1996) when the piezoelectric materials are used for both sensing and actuating to meet the collocation requirement. In fact, many researchers have considered a sensor bonded at the top surface and an actuator bonded at the same location but on the bottom surface of the host plates as being collocated (Tzou, 1993; Hwang and Park, 1993; Hwang et al., 1994; Lam et al., 1997; Liu et al., 1999; Lim et al., 1999; Kekana, 2002). This is actually different from the perfect collocation requirement and the question of whether such imperfect collocation, which leads to an asymmetric active damping matrix (Lim et al., 1999), can guarantee the dynamic stability has not been properly answered. This dynamic instability issue was first numerically reported by Wang et al. (2001), in which a continuously distributed, imperfectly collocated piezoelectric sensor/actuator pair on a cantilevered composite plate induces some in-plane displacements causing the control voltage to diverge after some time. However, no rigorous theoretical explanation to this phenomenon and practical remedy to design stable control systems were provided.

The objective of this paper is to present the dynamic stability analysis of active vibration control of piezoelectric composite plates using the negative-velocity feedback control law. By using the formulation in

(Wang et al., 2001), the stability analysis is performed based on a proposed Lyapunov's energy functional and the active damping matrix is examined in order to guarantee the dynamic stability. The significance of the in-plane displacements of the midplane with respect to dynamic stability is highlighted. To further confirm the results, the complex eigenvalues of the transformed linear first-order state space equations of motion will be examined. As complex eigensolution of large systems can be a formidable task, a stable state feedback law that satisfies the second Lyapunov's stability criteria strictly is proposed. The importance and effectiveness of the dynamics stability analysis will be demonstrated numerically.

2. Formulation of the piezoelectric FE model

The governing equations of motion for a piezoelectric composite plate were formulated in (Wang et al., 2001), assuming that: (a) the composite plate is thin or moderately thick and piezoelectric sensors/actuators are integrated into the laminated composite substrate as plies; (b) the laminate is perfectly bonded, elastic and orthotropic in behavior (Kekana, 2002) with small strains and displacements; (c) piezoelectric sensors/actuators are made of homogenous and isotropic dielectric materials (Cheng, 1989) and high electric fields and cyclic fields are not involved (Ehlers and Weisshaar, 1990).

Based on these assumptions, a linear constitutive relationship is adopted for the piezoelectric composite plate, which can be expressed as (Tzou and Tseng, 1990)

$$\boldsymbol{\sigma} = \mathbf{C}\varepsilon - \mathbf{e}^{\mathrm{T}}\mathbf{E}, \tag{1}$$

$$\mathbf{D} = \mathbf{e}\varepsilon + \mathbf{g}\mathbf{E}, \tag{2}$$

where $\boldsymbol{\sigma}$ represents the stress vector, $\mathbf{C}$ the elasticity matrix, ε the strain vector, $\mathbf{e}$ the piezoelectric constant matrix, $\mathbf{E}$ the electric field vector, $\mathbf{D}$ the electric displacement vector, and $\mathbf{g}$ the dielectric constant matrix. The magnetically static electric field vector $\mathbf{E}$ in the LPCE is related to the electric potential vector ϕ by using a gradient vector ∇ as

$$\mathbf{E} = -\nabla\phi. \tag{3}$$

Using Hamilton's variational principle (Hwang and Park, 1993), Wang (2002) derived the governing equations of motion of the piezoelectric composite plate in terms of the global coordinates with the standard procedure of the FE method as follows:

$$\begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\widetilde{\mathbf{U}}} \\ \ddot{\widetilde{\mathbf{\Phi}}}_s \\ \ddot{\widetilde{\mathbf{\Phi}}}_a \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{us} & \mathbf{K}_{ua} \\ \mathbf{K}_{su} & -\mathbf{K}_{ss} & \mathbf{0} \\ \mathbf{K}_{au} & \mathbf{0} & -\mathbf{K}_{aa} \end{bmatrix} \begin{Bmatrix} \widetilde{\mathbf{U}} \\ \widetilde{\mathbf{\Phi}}_s \\ \widetilde{\mathbf{\Phi}}_a \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{0} \\ \mathbf{Q} \end{Bmatrix}, \tag{4}$$

where $\widetilde{\mathbf{U}}$, $\widetilde{\mathbf{\Phi}}_s$ and $\widetilde{\mathbf{\Phi}}_a$ are the global coordinates representing the global generalized mechanical displacements, the global electric potentials of the piezoelectric sensors and actuators, respectively, $\mathbf{M}_{uu}$ the consistent mass matrix, $\mathbf{K}_{uu}$ the global stiffness matrix, $\mathbf{K}_{us}$ and $\mathbf{K}_{ua}$ the global electromechanical coupling matrix of the piezoelectric sensors and actuators, respectively, $\mathbf{K}_{ss}$ and $\mathbf{K}_{aa}$ the global dielectric stiffness matrices of the piezoelectric sensors and actuators (Samanta et al., 1996), respectively, $\mathbf{F}$ the global external mechanical forces, and $\mathbf{Q}$ the global external surface charges of the piezoelectric actuators (Wang, 2002). These matrices are defined in Appendix A. By condensing the potentials of the piezoelectric sensors in Eq. (4), the governing equations of motion can be re-written as

$$\mathbf{M}_{uu}\ddot{\widetilde{\mathbf{U}}} + \mathbf{K}_0\widetilde{\mathbf{U}} = \mathbf{F} + \mathbf{K}_{ua}\widetilde{\mathbf{\Phi}}_a, \tag{5}$$

where

$$\mathbf{K}_0 = \mathbf{K}_{uu} + \mathbf{K}_{us}\mathbf{K}_{ss}^{-1}\mathbf{K}_{su}. \tag{6}$$

The sensor output can be obtained in an alternative way by using the charge output of each electroplated sensor (Hanagud et al., 1992; Hwang and Park, 1993; Hwang et al., 1994; Samanta et al., 1996; Kwon and Bang, 1997; Lam et al., 1997; Saravanos et al., 1997; Lam and Ng, 1999; Wang et al., 2001; Wang, 2002). According to Eq. (4), the output electric charge vector $\mathbf{Q}_s$ of the piezoelectric sensors can be written (Wang et al., 2001; Wang, 2002) as

$$\mathbf{Q}_s = \mathbf{K}_{su}\widetilde{\mathbf{U}} \tag{7}$$

and thus the output voltage vector $\mathbf{\Phi}_c$ of the charge amplifiers can be obtained by choosing a proper electric circuit (Samanta et al., 1996; Hanagud et al., 1992) as

$$\mathbf{\Phi}_c = \mathbf{G}_c\frac{d\mathbf{Q}_s}{dt} = \mathbf{G}_c\mathbf{K}_{su}\dot{\widetilde{\mathbf{U}}}, \tag{8}$$

where $\mathbf{G}_c$ is a diagonal matrix of the constant gains of the corresponding charge amplifiers. Based on the charge output of the piezoelectric electroplated sensors, a negative-velocity feedback control system can be established (Hanagud et al., 1992; Hwang and Park, 1993; Hwang et al., 1994; Samanta et al., 1996; Lam et al., 1997; Kwon and Bang, 1997; Saravanos et al., 1997; Lam and Ng, 1999; Liu et al., 1999; Wang et al., 2001; Wang, 2002). In this output feedback control system, a constant symmetric gain matrix $\mathbf{G}$ (usually diagonal) is used to couple the output of the piezoelectric sensors and the input of the piezoelectric actuators with a change of the sign of the polarity as

$$\widetilde{\mathbf{\Phi}}_a = -\mathbf{G}\mathbf{\Phi}_c. \tag{9}$$

Substituting Eq. (8) into Eq. (9) yields

$$\widetilde{\mathbf{\Phi}}_a = -\mathbf{G}_0\mathbf{K}_{su}\dot{\widetilde{\mathbf{U}}}, \tag{10}$$

where $\mathbf{G}_0 = \mathbf{G}\mathbf{G}_c$ is the symmetric generalized gain matrix. Substituting Eq. (10) into Eq. (5), the governing equations of motion with active damping can be obtained as

$$\mathbf{M}_{uu}\ddot{\widetilde{\mathbf{U}}} + \mathbf{C}_A\dot{\widetilde{\mathbf{U}}} + \mathbf{K}_0\widetilde{\mathbf{U}} = \mathbf{F}, \tag{11}$$

where the piezoelectric active damping matrix $\mathbf{C}_A$, which is the contribution of the rate feedback (Hanagud et al., 1992), is given by

$$\mathbf{C}_A = \mathbf{K}_{ua}\mathbf{G}_0\mathbf{K}_{su}. \tag{12}$$

3. Dynamic stability analysis

Dynamic stability is of crucial importance for a controlled dynamic system (Kwon and Bang, 1997; Janocha, 1999). Unstable systems cannot be put into use in practice. The notion of dynamic stability implies that, after a bounded disturbance, the state variables of the system remain bounded, i.e. they stay within a defined space around a selected state (or even approach this state asymptotically). In this study, the second Lyapunov's stability theory is used to investigate the stability of the output feedback control and to design an unconditionally stable robust controller and the complex eigensolution method is used to examine the stability of those systems of which the dynamic stability cannot be guaranteed by the second Lyapunov's stability criteria.

3.1. Lyapunov's stability criteria

By pre-multiplying Eq. (11) by $(\dot{\widetilde{\mathbf{U}}})^{\mathrm{T}}$ and integrating it with respect to the time t and omitting the external forces $\mathbf{F}$, which can be considered as bounded (Kekana, 2002; Kwon and Bang, 1997) and thus does not affect the stability analysis, an energy flux equation is obtained as follows:

$$\int_t \dot{\widetilde{\mathbf{U}}}^{\mathrm{T}} \mathbf{M}_{uu} \, \mathrm{d}\dot{\widetilde{\mathbf{U}}} + \int_t \widetilde{\mathbf{U}}^{\mathrm{T}} \mathbf{K}_0 \, \mathrm{d}\widetilde{\mathbf{U}} + \int_t \dot{\widetilde{\mathbf{U}}}^{\mathrm{T}} \mathbf{C}_{\mathrm{A}} \dot{\widetilde{\mathbf{U}}} \, \mathrm{d}t = 0 \tag{13}$$

and thus a Lyapunov's energy functional $\mathscr{L}$ may be taken as

$$\mathscr{L} = \int_t \dot{\widetilde{\mathbf{U}}}^{\mathrm{T}} \mathbf{M}_{uu} \, \mathrm{d}\dot{\widetilde{\mathbf{U}}} + \int_t \widetilde{\mathbf{U}}^{\mathrm{T}} \mathbf{K}_0 \, \mathrm{d}\widetilde{\mathbf{U}} = \frac{1}{2}\left(\dot{\widetilde{\mathbf{U}}}^{\mathrm{T}} \mathbf{M}_{uu} \dot{\widetilde{\mathbf{U}}} + \widetilde{\mathbf{U}}^{\mathrm{T}} \mathbf{K}_0 \widetilde{\mathbf{U}}\right). \tag{14}$$

Substituting Eq. (13) into Eq. (14) yields

$$\mathscr{L} = -\int_t \dot{\widetilde{\mathbf{U}}}^{\mathrm{T}} \mathbf{C}_{\mathrm{A}} \dot{\widetilde{\mathbf{U}}} \, \mathrm{d}t. \tag{15}$$

Hence, the time derivative of the Lyapunov's energy functional $\mathscr{L}$ is:

$$\dot{\mathscr{L}} = -\dot{\widetilde{\mathbf{U}}}^{\mathrm{T}} \mathbf{C}_{\mathrm{A}} \dot{\widetilde{\mathbf{U}}}. \tag{16}$$

According to the second Lyapunov's stability theory (Kwon and Bang, 1997), the dynamic stability can be guaranteed if $\dot{\mathscr{L}} \leqslant 0$ and thus $\mathbf{C}_{\mathrm{A}}$ must be positive semi-definite. Furthermore, asymptotic stability can be guaranteed if $\dot{\mathscr{L}}$ remains negative except for the equilibrium point ($\widetilde{\mathbf{U}} = \mathbf{0}$) and thus $\mathbf{C}_{\mathrm{A}}$ must be positive definite. Since the Lyapunov's functional $\mathscr{L}$ is based on the energy of the system, its time derivative $\dot{\mathscr{L}}$ in Eq. (16) represents the energy flux of the system. According to the active damping matrix in Eq. (12), it can be inferred that for an asymptotically stable system, to increase the control gain $\mathbf{G}$ leads to the $\dot{\mathscr{L}}$ larger in magnitude and negative in sign so that the system energy can be extracted more rapidly. Therefore, for an asymptotically stable system, the higher the control gain, the faster the decay of the vibration. Based on the Lyapunov's stability criteria, the dynamic stability of the output feedback control system in Eq. (11) is to be investigated next.

3.2. Dynamic stability of the output feedback control system

If each piece of piezoelectric material is used for both sensing and actuating, then each piezoelectric sensor/actuator pair can be regarded as perfectly collocated (Chen et al., 1996). In this case, we have

$$\mathbf{K}_{su} = \mathbf{K}_{au} = \mathbf{K}_{ua}^{\mathrm{T}}. \tag{17}$$

Substituting Eq. (17) into Eq. (12) yields

$$\mathbf{C}_{\mathrm{A}} = \mathbf{K}_{ua} \mathbf{G}_0 \mathbf{K}_{ua}^{\mathrm{T}}. \tag{18}$$

Hence, $\mathbf{C}_{\mathrm{A}}$ will be positive semi-definite in general or positive definite if $\mathbf{K}_{ua}^{\mathrm{T}}$ is an $n \times m$ matrix of rank $n < m$. Therefore, the dynamic stability can be guaranteed and furthermore, asymptotic stability can also be guaranteed if the dynamic characteristics of the self-sensing actuators are properly selected. This is consistent with the stability requirement of the classical negative-velocity feedback control that the sensors and actuators must be collocated and have perfect dynamic properties (Preumont, 1997).

However, significant non-linearities would exist in this controlled system due to the interactions between actuating and sensing signals (Chen et al., 1996). To minimize such non-linearities,, in practice, two separate, but very close to each other, pieces of piezoelectrics are often used and even considered as an approximate collocation (Ang et al., 2002; Wang, 2002). As a further approximation, piezoelectric sensor/

actuator pairs bonded at the top and bottom surfaces of the host structure, respectively, have also been considered as collocated by many researchers in their models (Hanagud et al., 1992; Hwang and Park, 1993; Hwang et al., 1994; Samanta et al., 1996; Lam et al., 1997; Saravanos et al., 1997; Lam and Ng, 1999; Liu et al., 1999; Kekana, 2002). Stability investigation of such approximations is therefore of practical importance.

According to Eq. (12), it can be concluded that in general, the active damping matrix $\mathbf{C}_\mathrm{A}$ will not be positive semi-definite and thus the dynamic stability cannot be guaranteed if the piezoelectric sensor/actuator pairs are not perfectly collocated. As a demonstration, investigation into the active damping matrix $\mathbf{C}_\mathrm{A}$ is performed on a dynamic system with a single piezoelectric PVDF or PZT sensor/actuator pair bonded or embedded along the thickness of a composite plate. As shown in (Wang et al., 2001; Wang, 2002), the active damping matrix $\mathbf{C}_\mathrm{A}$ for a typical piezoelectric quadratic isoparametric element can be written as

$$C_\mathrm{A} = G_0[f_1^a \quad f_2^a \quad \cdots \quad f_8^a]^\mathrm{T}[k_1^s \quad k_2^s \quad \cdots \quad k_8^s], \tag{19}$$

where the product of the diagonal submatrices $\{f_l^a\}[k_l^s](l = 1, 2, \ldots, 8)$ is given by

$$\{f_l^a\}[k_l^s] = \begin{bmatrix} (A_l^x)^2 & A_l^x A_l^y & 0 & z_0^s (A_l^x)^2 & z_0^s A_l^x A_l^y \\ A_l^x A_l^y & (A_l^y)^2 & 0 & z_0^s A_l^x A_l^y & z_0^s (A_l^y)^2 \\ 0 & 0 & 0 & 0 & 0 \\ z_0^a (A_l^x)^2 & z_0^a A_l^x A_l^y & 0 & z_0^s z_0^a (A_l^x)^2 & z_0^s z_0^a A_l^x A_l^y \\ z_0^a A_l^x A_l^y & z_0^a (A_l^y)^2 & 0 & z_0^s z_0^a A_l^x A_l^y & z_0^s z_0^a (A_l^y)^2 \end{bmatrix}, \tag{20}$$

in which f_l^a, k_l^s, A_l^x and A_l^y are defined in Appendix B z_0^s and z_0^a the coordinates of the sensor and actuator along the thickness direction, respectively.

If the sensor and actuator are bonded at the top and bottom surfaces of the composite plate, respectively, Eq. (20) turns out to be

$$\{f_l^a\}[k_l^s] = \begin{bmatrix} (A_l^x)^2 & A_l^x A_l^y & 0 & -z_0^a (A_l^x)^2 & -z_0^a A_l^x A_l^y \\ A_l^x A_l^y & (A_l^y)^2 & 0 & -z_0^a A_l^x A_l^y & -z_0^a (A_l^y)^2 \\ 0 & 0 & 0 & 0 & 0 \\ z_0^a (A_l^x)^2 & z_0^a A_l^x A_l^y & 0 & (z_0^a A_l^x)^2 & (z_0^a)^2 A_l^x A_l^y \\ z_0^a A_l^x A_l^y & z_0^a (A_l^y)^2 & 0 & (z_0^a)^2 A_l^x A_l^y & (z_0^a A_l^y)^2 \end{bmatrix}. \tag{21}$$

It can be inferred from Eq. (20) that the diagonal submatrices of the active damping matrix are not symmetric due to the so-called asymmetric stretching–bending coupling effect of the piezoelectric sensor/actuator pair (Wang et al., 2001) and thus the active damping matrix will not be positive semi-definite. Therefore, the dynamic stability of this feedback control system, as used in (Hanagud et al., 1992; Hwang and Park, 1993; Hwang et al., 1994; Samanta et al., 1996; Lam et al., 1997; Saravanos et al., 1997; Lam and Ng, 1999; Liu et al., 1999), cannot be guaranteed.

Furthermore, if the corresponding in-plane displacements of the midplane of the plate are not allowed by the boundary conditions, such as in the cases of simply-supported plates or clamped plates, the product of the diagonal submatrices in Eq. (20) reduces to symmetric submatrices given by

$$\{f_l^a\}[k_l^s] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & z_0^s z_0^a (A_l^x)^2 & z_0^s z_0^a A_l^x A_l^y \\ 0 & z_0^s z_0^a A_l^x A_l^y & z_0^s z_0^a (A_l^y)^2 \end{bmatrix} \tag{22}$$

and the corresponding active damping matrix $\mathbf{C}_\mathrm{A}$ will be positive semi-definite. Hence, the dynamic stability can be guaranteed. Similarly, if the in-plane displacements of the midplane of the plate are omitted for the purpose of simplicity, which was assumed by most of the analytical models such as in (Hanagud et al.,

1992) and some of finite element models such as in (Hwang and Park, 1993; Hwang et al., 1994), the dynamic stability of this simplified dynamic system can also be guaranteed, however, according to the present study, this conclusion is not justified since it does not hold true for the original system without this simplification and it can be expected that significant errors will be caused if the original dynamic system is actually unstable, as demonstrated later in Section 4.

Therefore, the dynamic stability of an output feedback control system using non-perfectly collocated sensor/actuator pairs cannot be guaranteed. To ensure the dynamic stability of such a system, strict stability analysis becomes necessary and thus a complex eigensolution method is next introduced.

3.3. Complex eigensolution method

The governing equations of motion Eq. (5) in the absence of exogenous disturbance ($\mathbf{F} = \mathbf{0}$) and the sensor equation (8) can be transformed into the first-order linear state space form as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{23}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \tag{24}$$

where $\mathbf{x} = [\widetilde{\mathbf{U}}\dot{\widetilde{\mathbf{U}}}]^{\mathrm{T}}$ is the state variable vector, $\mathbf{y} = \mathbf{\Phi}_{\mathrm{c}}$ the output voltage vector, $\mathbf{u} = \widetilde{\mathbf{\Phi}}_a$ the actuator input voltage vector, and the system matrix $\mathbf{A}$, input matrix $\mathbf{B}$ and output matrix $\mathbf{C}$ are given in Appendix B. Hence, the output feedback controller in Eq. (9) can be expressed as

$$\mathbf{u} = -\mathbf{G}\mathbf{C}\mathbf{x}. \tag{25}$$

Substituting Eq. (25) into Eq. (23) yields

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{G}\mathbf{C})\mathbf{x} \tag{26}$$

and thus a characteristic equation is obtained as follows:

$$|\lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{G}\mathbf{C})| = 0, \tag{27}$$

to which the solution λ is actually the complex eigenvalue of the $(\mathbf{A} - \mathbf{B}\mathbf{G}\mathbf{C})$ matrix. According to the dynamic stability theory (Kwon and Bang, 1997), a sufficient and necessary condition for the dynamic stability is that all the complex eigenvalues are with non-positive real parts. Furthermore, a sufficient and necessary condition for the asymptotic stability is that all the complex eigenvalues are with negative real parts.

Hence, the determination of all the complex eigenvalues would play a central role in the dynamic stability analysis. However, the procedure to calculate all the complex eigenvalues of Eq. (27) is prohibitively expensive and numerically formidable for large problems in general. In fact, this numerical calculation procedure is not needed for a feedback control system that satisfies the Lyapunov's second stability criteria strictly (Kekana, 2002; Kwon and Bang, 1997). Therefore, such a robust controller would be more desirable in practice and is thus presented next.

3.4. Robust feedback controller design

A negative-velocity feedback controller may be taken as

$$\widetilde{\mathbf{\Phi}}_a = -\mathbf{G}_0\mathbf{K}_{au}\dot{\widetilde{\mathbf{U}}}. \tag{28}$$

This is one of the simple ways to design a robust controller. Comparing Eq. (28) with Eq. (10), it can be found that this is a state feedback controller, rather than an output feedback controller, since the output of the piezoelectric sensors $\mathbf{\Phi}_{\mathrm{c}}$ is not used directly to construct the input vector $\widetilde{\mathbf{\Phi}}_a$. However, if the paired sensors and actuators are bonded together, it can be approximated that $\mathbf{K}_{au} = \mathbf{K}_{su}$ since the thickness of the piezoelectric layers is usually two to three orders less than that of the substrate (Samanta et al., 1996), and in this case the output of the piezoelectric sensors can be used to construct a stable controller. More generally, in order to achieve this feedback control, all the feedback control states $\tilde{\mathbf{U}}$ must be accessible and thus the dynamic system must be observable.

Substituting Eq. (28) into Eq. (5), the active damping matrix can be obtained as

$$\mathbf{C}_{\mathrm{A}} = \mathbf{K}_{ua}\mathbf{G}_0\mathbf{K}_{au}. \tag{29}$$

Hence, $\mathbf{C}_{\mathrm{A}}$ will be positive semi-definite in general or positive definite if $\mathbf{K}_{au}$ is an $n \times m$ matrix of rank $n < m$. Therefore, the dynamic stability can be guaranteed and furthermore, asymptotic stability can also be guaranteed if the piezoelectric actuators are properly configured.

4. Numerical results and discussion

The numerical example used in (Lam et al., 1997) is adopted to demonstrate the importance of the present dynamic stability analysis and effectiveness of the present design strategies of stable systems. As shown in Fig. 1, the piezoelectric composite plate (Lam et al., 1997) consists of four composite substrate layers and two outer PZT layers bonded at the top and bottom surfaces of the substrate serving as sensor and actuator, respectively. The stacking sequence of the substrate is antisymmetric angle-ply [−45°/45°/−45°/45°]. The substrate is made of T300/976 graphite-epoxy composite and the piezoelectric layers are made of PZT G1195N and their corresponding material properties can be found in (Lam et al., 1997). It is assumed that the vibration of the plate is excited by a suddenly removed vertical load 1N initially applied at the tip point A, which is in the midplane of the plate.

4.1. Unstable negative-velocity feedback control

The time histories of the tip deflection at point A and the input voltage of the piezoelectric actuator when using the output feedback control law in Eq. (9) are shown in Figs. 2 and 3, respectively. As expected, when the control gain is zero, the input voltage of the actuator will be zero too so that the free vibration will not decay since the passive structural damping is not considered in the present study. Increasing the generalized control gain leads to the divergence in the response of the tip deflection and more significantly the input voltage. This illustrates the instability of the feedback control system. This conclusion can be further

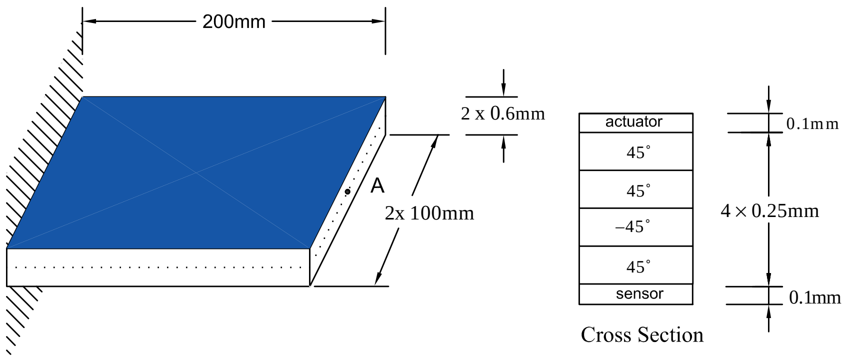

Fig. 1. The geometry of a piezoelectric composite plate.

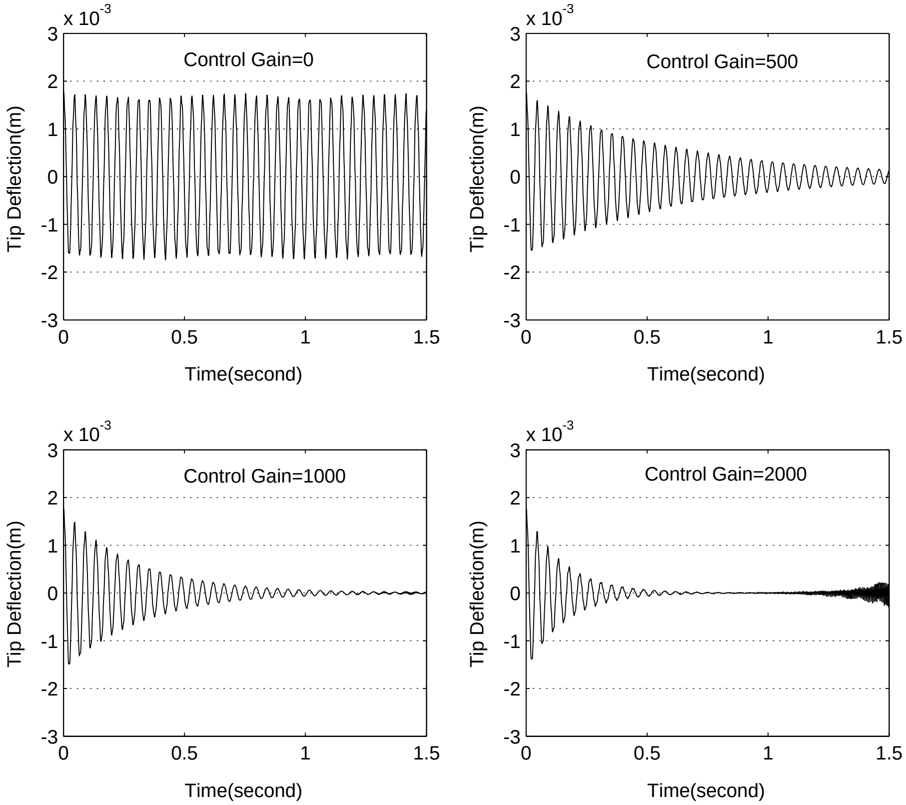

Fig. 2. Time history of the deflection of the tip point *A* of the piezoelectric composite plate.

verified by computing the complex eigenvalues in Eq. (27). Table 1 gives only the complex eigenvalues of the first five modes of this dynamic system. Note that the second and fifth complex eigenvalues are always with positive real part when the control gain is not zero, which verifies that the dynamic system is not stable. This phenomenon can also be partially attributed to the fact that bonding the sensor/actuator pairs at the top and bottom surfaces, respectively, is not a sufficient condition for the dynamic stability, as discussed in detail in Section 3.2. This phenomenon was not reported by Lam et al. (1997) or an earlier work by Tzou (1993), since both of them changed their output feedback control law into a tip point velocity feedback law to predict the response. The latter is different from their formulations based on the output feedback control law and adds the extra requirement of introducing a point velocity sensor.

In the analytical model of Hanagud et al. (1992) and the finite element model of Hwang and Park (1993) and Hwang et al. (1994), in-plane displacements of the midplane have been neglected. By adopting this simplification, the time histories of the tip deflection at point *A* and the input voltage of the piezoelectric actuator of this altered dynamic system are shown in Figs. 4 and 5, respectively. It can be seen that the dynamic response will decay with a moderate control gain and thus the results suggest that the corresponding dynamic system is asymptotically stable, consistent with the prediction in Section 3.2. Considering the fact that the original dynamic system is actually not stable, it can be concluded that significant numerical errors are caused by adopting this simplification. Therefore, generally, omitting the in-plane displacements of the midplane of the composite plate cannot be justified in the dynamic stability analysis.

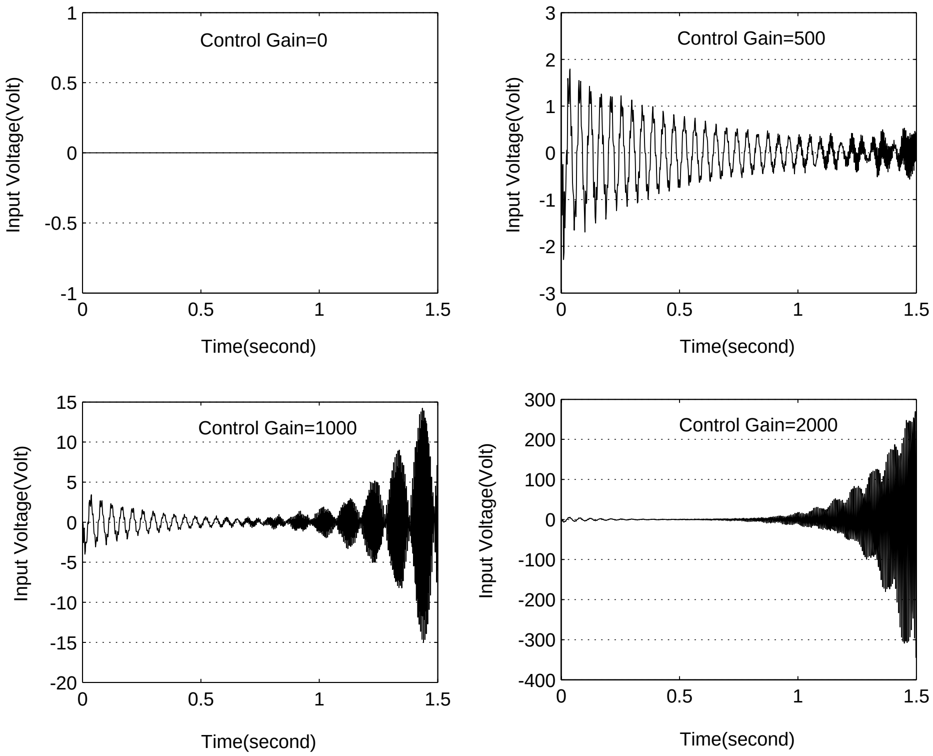

Fig. 3. Time history of the input voltage of the actuator of the piezoelectric composite plate.

Table 1
Complex eigenvalues of the piezoelectric composite plate

Gains	0	500	1000	2000
λ_1	142.21i	−1.64+142.25i	−3.27+142.37i	−6.53+142.83i
λ_2	403.14i	0.92+403.06i	1.79+402.83i	3.3+402.02i
λ_3	844.06i	−2.06+844.59i	−3.49+845.85i	−4.29+848.49i
λ_4	1204.22i	−63.77+1212.44i	−124.38+1238.89i	−195.94+1346.13i
λ_5	1394.49i	0.84+1397.63i	1.56+1397.99i	2.44+1399.02i

4.2. Stable negative-velocity feedback control systems by adjusting the configurations

To overcome dynamic instability, the system has to be re-designed. Without changing the output feedback control law in Eq. (9), one alternative design is to use an approximately collocated sensor/actuator pair by bonding the actuator layer on the bottom surface of the sensor layer, as shown in Fig. 6. The time histories of the tip deflection at point A and the input voltage of the piezoelectric actuator of this altered dynamic system are shown in Figs. 7 and 8, respectively. When the control gain is zero, the peak values of tip deflection of this system are a bit larger than those of the original system shown in Fig. 2, due to the dominant reduction of the bending stiffness of the piezoelectric sensor layer with respect to the shifted new midplane. This observation can be further verified by the differences of the natural frequencies shown in Tables 1 and 2 when the corresponding control gain is zero. It can be seen that the natural frequencies of

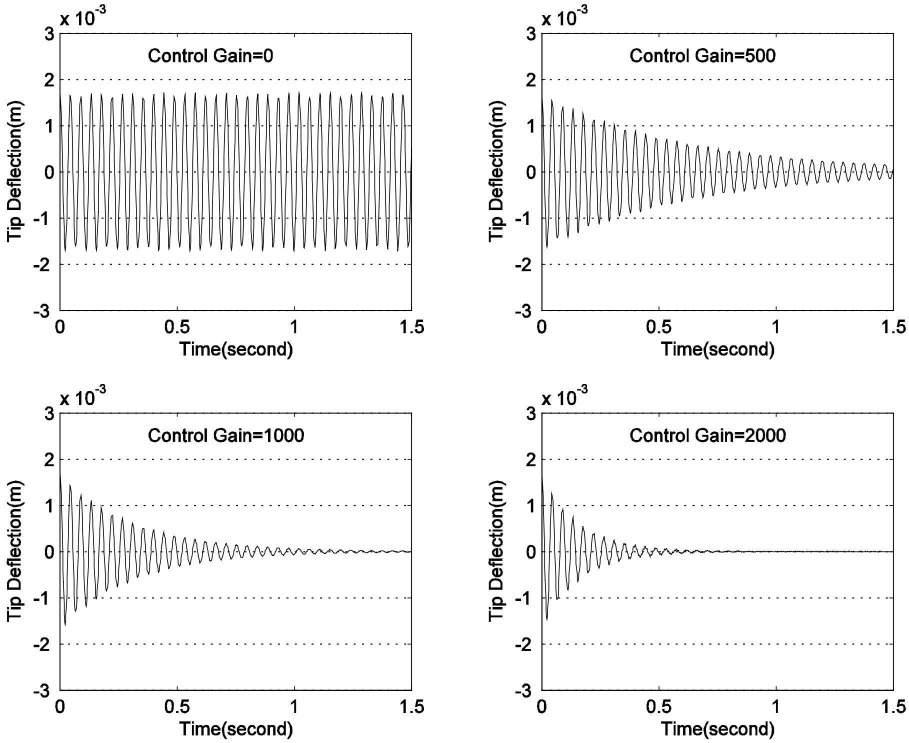

Fig. 4. Time history of the deflection of the tip point A of the piezoelectric composite plate without in-plane displacements.

this altered system are smaller than those of the original system, which suggests that a stiffness reduction has been caused by this alteration. It can also be seen that the dynamic system will return to its equilibrium point ($\dot{\tilde{\mathbf{U}}} = \mathbf{0}$) in about 1.5 s with the moderate control gains 500, 1000 and 2000. Higher control gain results in faster decay of the dynamic response. This suggests that the altered dynamic system is asymptotically stable since the sensor/actuator pair is approximately collocated. Further verification of the dynamic stability can be done by finding the complex eigenvalues in Eq. (27). Table 2 only presents the complex eigenvalues of the first five modes of this altered dynamic system. All the complex eigenvalues are with negative real parts and thus the asymptotic stability is guaranteed.

Without changing the output feedback control law in Eq. (9), another alternative design is to use a symmetric, rather than antisymmetric, angle-ply composite laminate [−45°/45°/45°/−45°], as shown in Fig. 9. The time histories of the tip deflection at point A and the input voltage of the piezoelectric actuator of this altered dynamic system are shown in Figs. 10 and 11, respectively. When the control gain is zero, the peak values of tip deflection of this system are similar to those of the original system shown in Fig. 2, due to the fact that neither the additional stretching–bending coupling effect of the antisymmetric balanced regular angle-ply composite substrate [−45°/45°/−45°/45°] nor the additional bending–twisting coupling effect of the symmetric balanced angle-ply composite substrate [−45°/45°/45°/−45°] is significant (Reddy, 1997). This observation can be further verified by the similarities of the natural frequencies shown in Tables 3 and 2 when the corresponding control gain is zero. It can be seen that the natural frequencies of this altered system are close to those of the original system and the maximum error is 2.1%, which suggests that this alteration does not affect the stiffness significantly. It can also be seen that the dynamic system will return to its equilibrium point ($\tilde{\mathbf{U}} = \mathbf{0}$) in about 1.5 s with moderate control gains. It suggests that this altered

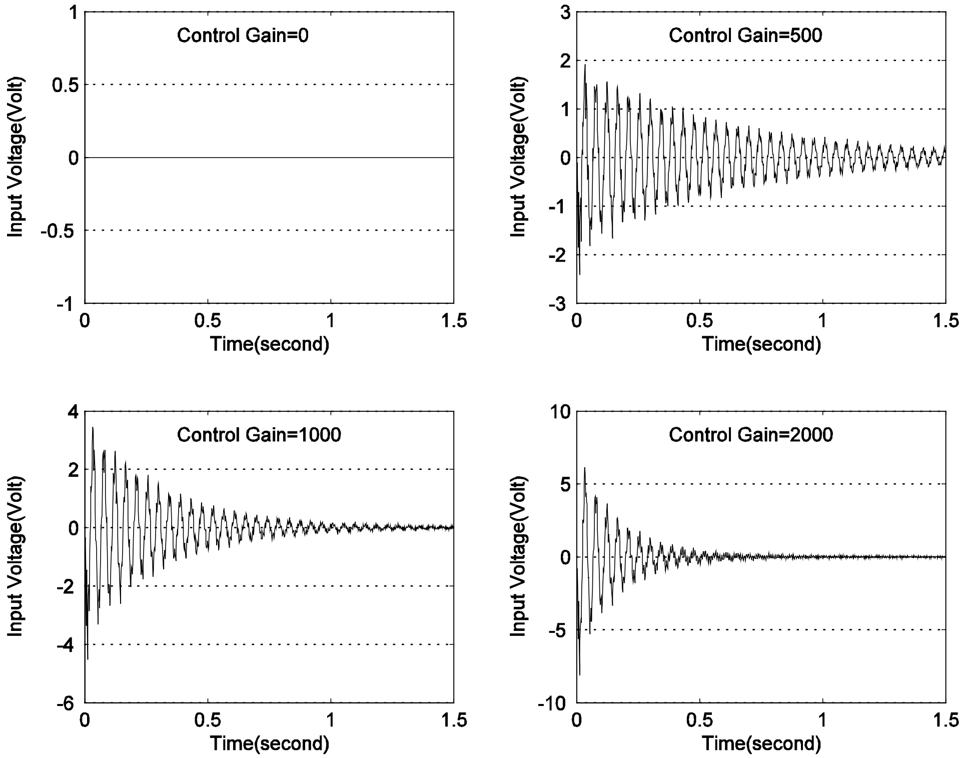

Fig. 5. Time history of the input voltage of the actuator of the piezoelectric composite plate without in-plane displacements.

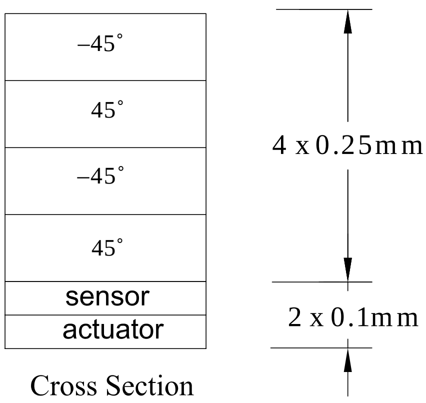

Fig. 6. Cross-section of the piezoelectric composite plate of the altered dynamic system with an imperfectly collocated sensor/actuator pair.

dynamic system is also asymptotically stable since the stretching–bending coupling effect of the composite laminate has been eliminated by using the symmetric angle-ply sequence and thus the in-plane displacements of the midplane would become negligible. It can be further verified by solving the complex eigenvalues in Eq. (27). Table 3 presents the complex eigenvalues of the first five modes of this altered dynamic

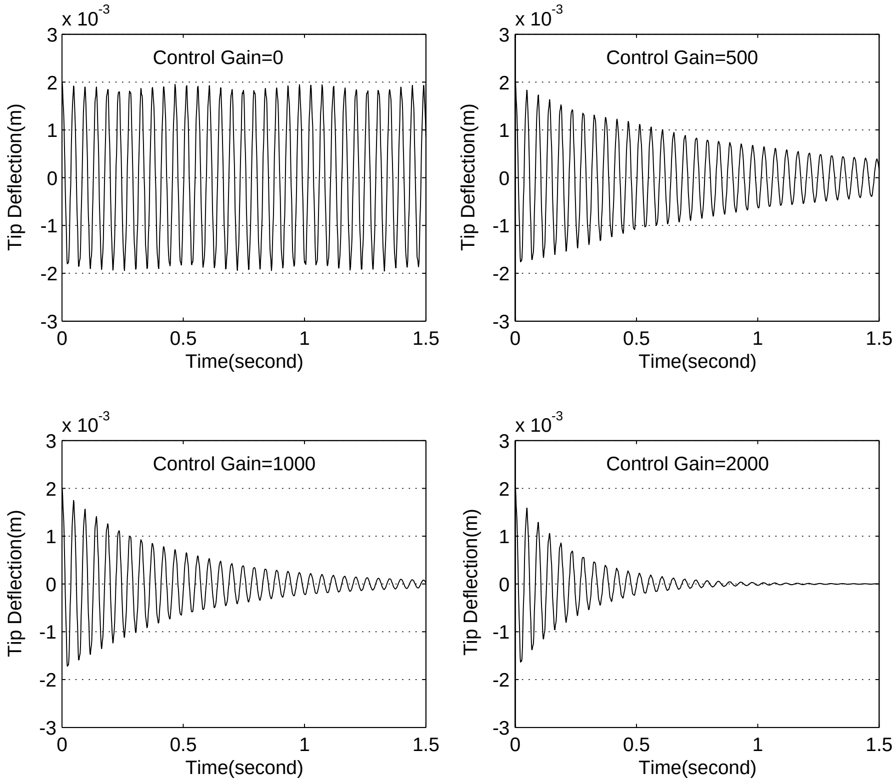

Fig. 7. Time history of the deflection of the tip point *A* of the altered dynamic system with an imperfectly collocated sensor/actuator pair.

Table 2
Complex eigenvalues of the altered dynamic system with an imperfectly collocated sensor/actuator pair

Gains	0	500	1000	2000
λ_1	134.93i	−1.06+134.95i	−2.11+135.01i	−4.21+135.26i
λ_2	399.24i	−0.23+399.25i	−0.46+399.29i	−0.88+399.46i
λ_3	797.28i	−1.44+797.53i	−2.64+798.2i	−3.97+800.09i
λ_4	1162.12i	−41.87+1166.86i	−82.52+1181.54i	−148.03+1243.97i
λ_5	1374.43i	−2.47+1374.27i	−4.91+1373.79i	−9.58+1371.77i

system, where all the complex eigenvalues are with negative real parts and thus the asymptotic stability can be guaranteed.

4.3. Stable negative-velocity feedback control based on the second Lyapunov's stability criteria

The above two altered dynamic system designs show that without changing the output feedback control law, the dynamic stability can also be achieved by either bonding the sensor/actuator pair together to construct an approximate collocation or changing the ply stacking sequence of the composite substrate to eliminate the stretching–bending coupling effect of the composite laminate. However, these two design

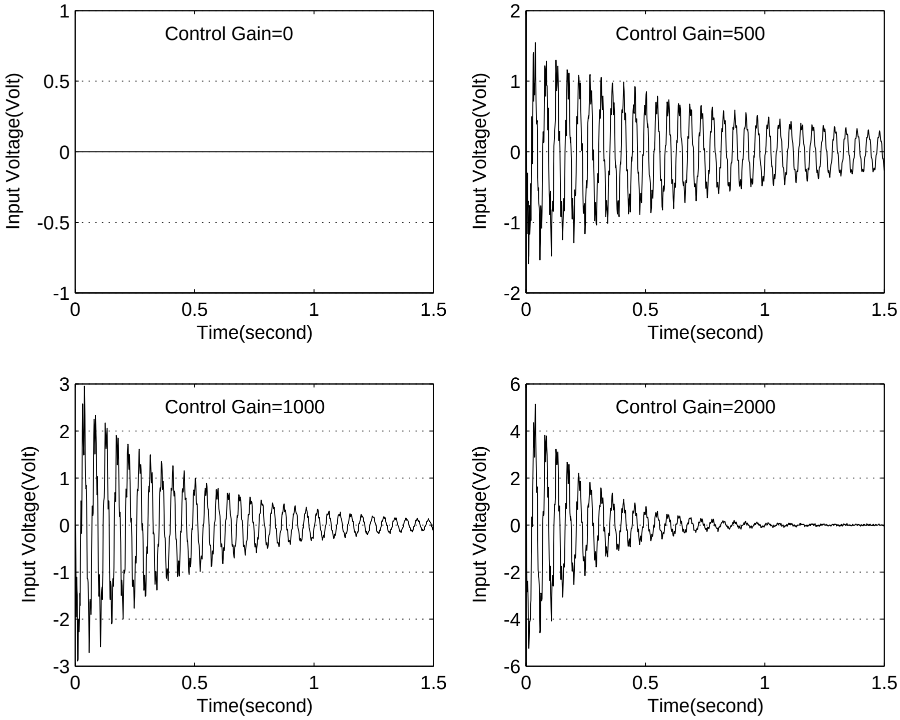

Fig. 8. Time history of the input voltage of the actuator of the altered dynamic system with an imperfectly collocated sensor/actuator pair.

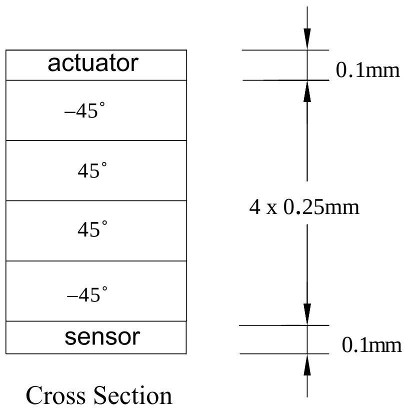

Fig. 9. Cross-section of the piezoelectric composite plate of the altered dynamic system with a symmetric angle-ply composite substrate.

methods may be problem dependent. Since Lyapunov's second stability criteria may not be completely satisfied, there is no guarantee that all the resulting dynamic systems based on these two methods be stable. More importantly, the design procedure would become computationally more expensive, if not impossible, for a larger system because of the need to calculate all the complex eigenvalues. To reduce the computa-

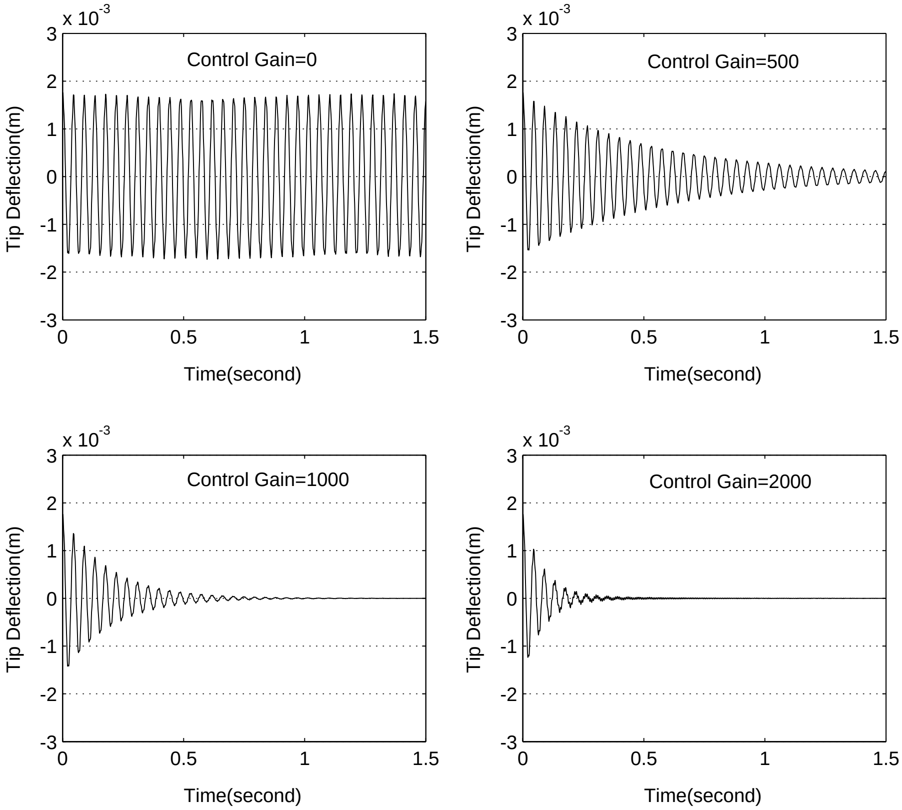

Fig. 10. Time history of the deflection of the tip point A of the dynamic system with a symmetric angle-ply composite substrate.

tional cost and/or to avoid the numerical difficulties involved, the proposed robust controller in Eq. (28) can be used for vibration control of the piezoelectric composite plate. The time histories of the tip deflection at point A and the input voltage of the piezoelectric actuator of this altered dynamic system are shown in Figs. 12 and 13, respectively. With moderate control gains this dynamic system will return to its equilibrium point in about 1.5 s. Higher control gain yields faster decay in the response. This observation also suggests that the dynamic system is stable, as confirmed by the complex eigenvalues shown in Table 4, in which only the first five eigenvalues are listed. As expected, when the control gain is zero, the natural frequencies of this altered dynamic system (Table 4) are the same as that of the original dynamic system (Table 1) since the difference between them is the control law only. For the demonstrative purpose only, it can be found that all the eigenvalues of this altered system are with negative real parts when the control gain is larger than zero. This system with a state feedback control law is therefore asymptotically stable.

5. Conclusions

This paper demonstrates the importance of using an appropriate FE model and an actually stable negative-velocity feedback control law to perform the vibration control of piezoelectric composite plates. Based on the second Lyapunov's stability criteria by using a proposed Lyapunov's energy functional, it is found that the imperfect collocation of piezoelectric sensor/actuator pairs is not sufficient for dynamic stability in general and that ignoring the in-plane displacements of the midplane of the composite plate may

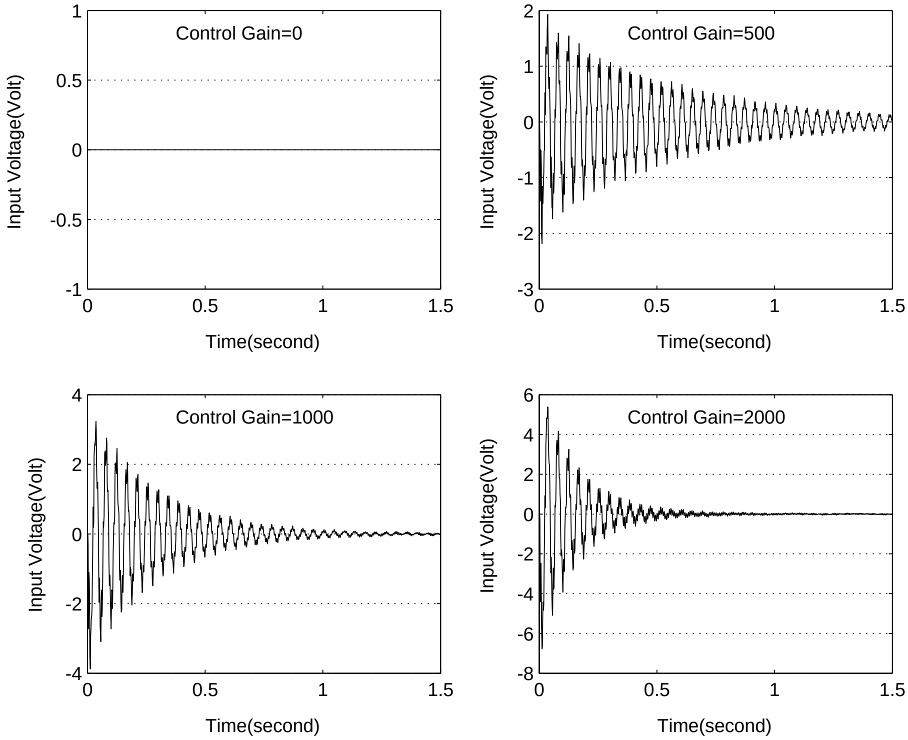

Fig. 11. Time history of the input voltage of the actuator of the dynamic system with a symmetric angle-ply composite substrate.

Table 3
Complex eigenvalues of the altered dynamic system with a symmetric angle-ply composite substrate

Gains	0	500	1000	2000
λ_1	142.46i	−1.77+142.51i	−3.55+142.66i	−7.06+143.26i
λ_2	398.31i	−1.5+398.43i	−2.95+398.79i	−5.5+400.1i
λ_3	849.47i	−1.68+849.91i	−2.82+850.95i	−3.39+853.08i
λ_4	1166.13i	−59.12+1173.92i	−116.23+1199.13i	−188.88+1307.31i
λ_5	1423.92i	−3.04+1423.73i	−6.08+1423.15i	−11.98+1420.64i

cause significant numerical errors, leading to incorrect stability conclusions. Using a proposed state feedback law and the Lyapunov's energy functional to perform stability analysis, dynamic stability can be guaranteed. This is a cost-efficient and effective alternative to performing a complex eigen-analysis especially for large systems. Theoretical considerations are confirmed by numerical results, where the signs of the real parts of all the complex eigenvalues are used to determine the dynamic stability. It is also shown numerically that asymptotically stable control systems may be obtained by bonding the piezoelectric sensor/actuator pairs together or changing the ply stacking sequence of the composite substrate.

Appendix A

As discussed in (Wang, 2002), $\mathbf{N}_u$ is the matrix of displacement shape functions; ρ the mass density; Ω_i the volume of ith element; $\mathbf{B}_u$ the strain-displacement matrix; $\mathbf{B}_\phi$ the electric field-potential matrix; $\mathbf{P}_\mathrm{b}$, $\mathbf{P}_\mathrm{s}$

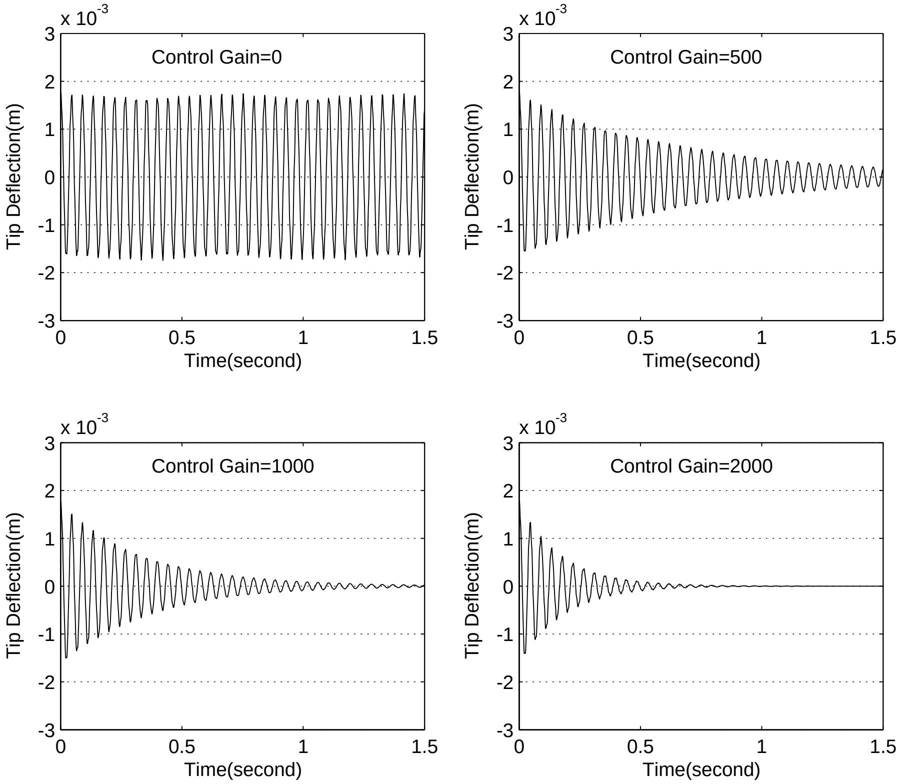

Fig. 12. Time history of the deflection of the tip point A of the piezoelectric composite plate using the robust controller.

and $\mathbf{P}_i$ the body, surface and ith concentrated load vectors, respectively; $\mathbf{q}$ the surface charge density vector; Γ_s the external mechanical loading surface, and Γ_ϕ the external electrical loading surface.

$$\mathbf{M}_{uu} = \sum_i \int_{\Omega_i} \mathbf{N}_u^{\mathrm{T}} \rho \mathbf{N}_u \,\mathrm{d}\Omega, \tag{A.1}$$

$$\mathbf{K}_{uu} = \sum_i \int_{\Omega_i} \mathbf{B}_u^{\mathrm{T}} \mathbf{C} \mathbf{B}_u \,\mathrm{d}\Omega, \tag{A.2}$$

$$\mathbf{B}_\phi = \begin{bmatrix} \mathbf{B}_{\phi s} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\phi a} \end{bmatrix}, \quad \mathbf{K}_{su} = \mathbf{K}_{us}^{\mathrm{T}}, \quad \mathbf{K}_{au} = \mathbf{K}_{ua}^{\mathrm{T}}, \tag{A.3}$$

$$\mathbf{K}_{us} = \sum_i \int_{\Omega_i} \mathbf{B}_u^{\mathrm{T}} \mathbf{e}^{\mathrm{T}} \mathbf{B}_{\phi s} \,\mathrm{d}\Omega, \tag{A.4}$$

$$\mathbf{K}_{ua} = \sum_i \int_{\Omega_i} \mathbf{B}_u^{\mathrm{T}} \mathbf{e}^{\mathrm{T}} \mathbf{B}_{\phi a} \,\mathrm{d}\Omega, \tag{A.5}$$

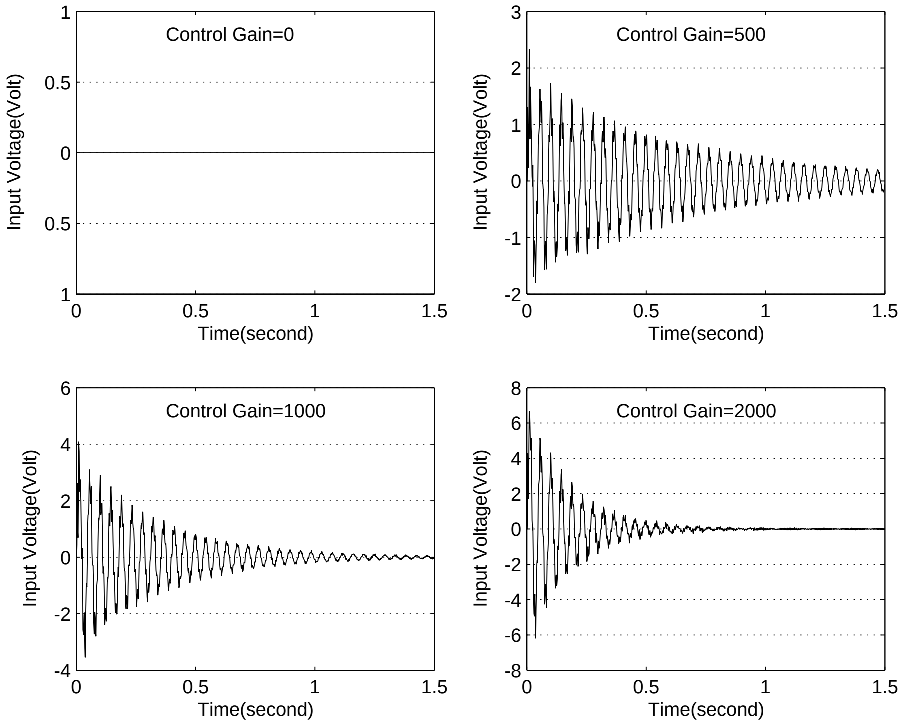

Fig. 13. Time history of the input voltage of the actuator of the piezoelectric composite plate using the robust controller.

Table 4
Complex eigenvalues of the altered dynamic system with a robust feedback controller

Gains	0	500	1000	2000
λ_1	142.21i	−1.44+142.24i	−2.88+142.35i	−5.74+142.77i
λ_2	403.14i	−0.76+403.19i	−1.5+403.36i	−2.82+403.98i
λ_3	844.06i	−1.8+844.46i	−3.16+845.47i	−4.22+847.85i
λ_4	1204.22i	−55.77+1210.99i	−109.95+1232.5i	−188.16+1325.1i
λ_5	1397.49i	−1.21+1397.32i	−2.3+1396.83i	−3.75+1295.26i

$$\mathbf{K}_{ss} = \sum_i \int_{\Omega_i} \mathbf{B}_{\phi s}^{\mathrm{T}} \mathbf{g} \mathbf{B}_{\phi s} \, \mathrm{d}\Omega, \tag{A.6}$$

$$\mathbf{K}_{aa} = \sum_i \int_{\Omega_i} \mathbf{B}_{\phi a}^{\mathrm{T}} \mathbf{g} \mathbf{B}_{\phi a} \, \mathrm{d}\Omega, \tag{A.7}$$

$$\mathbf{F} = \int_{\Omega} \mathbf{N}_u^{\mathrm{T}} \mathbf{P}_{\mathrm{b}} \, \mathrm{d}\Omega + \int_{\Gamma_{\mathrm{s}}} \mathbf{N}_u^{\mathrm{T}} \mathbf{P}_{\mathrm{s}} \, \mathrm{d}\Gamma_{\mathrm{s}} + \sum_i \mathbf{N}_{ui}^{\mathrm{T}} \mathbf{P}_i, \tag{A.8}$$

$$\mathbf{Q} = -\int_{\Gamma_\phi} \mathbf{N}_\phi^{\mathrm{T}} \mathbf{q} \, \mathrm{d}\Gamma_\phi. \tag{A.9}$$

Appendix B

$$A_l^x = e_{31}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial N_{ul}}{\partial x}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{B.1}$$

$$A_l^y = e_{32}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial N_{ul}}{\partial y}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{B.2}$$

in which $l = 1, 2, \ldots, 8$, e_{31} and e_{32} are the dielectric constants of the piezoelectric materials, x and y the spacial coordinates, ξ and η the natural coordinates (Wang, 2002).

$$\{f_l^a\} = [A_l^x \quad A_l^y \quad 0 \quad z_0^a A_l^x \quad z_0^a A_l^y]^T, \tag{B.3}$$

$$[k_l^s] = [A_l^x \quad A_l^y \quad 0 \quad z_0^s A_l^x \quad z_0^s A_l^y]. \tag{B.4}$$

The system matrix **A**, input matrix **B** and output matrix **C** are given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_{uu}^{-1}\mathbf{K}_0 & \mathbf{0} \end{bmatrix}, \tag{B.5}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_{uu}^{-1}\mathbf{K}_{ua} \end{bmatrix}, \tag{B.6}$$

$$\mathbf{C} = [\mathbf{0} \quad \mathbf{G}_c\mathbf{K}_{su}]. \tag{B.7}$$

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